Q1

1

Use an appropriate method to differentiate each of the following.

(i) $\tan 3x + e^{7-2x^2}$

(ii) $(x^2 + 2x - 8)\cos(3 - x)$

(iii) $\dfrac{\ln 7x}{\sin(x^2+5)}$

(iv) $\sqrt{\cos 4x}$

[8]

CHAIN RULE $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ | PRODUCT RULE $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ | QUOTIENT RULE $\dfrac{d\left(\frac{u}{v}\right)}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

i) $3\sec^2 3x + (-4x)e^{7-2x^2}$

$= \boxed{3\sec^2 3x - 4xe^{7-2x^2}}$

ii) let $u = x^2 + 2x - 8$ $\quad v = \cos(3-x)$

$\dfrac{du}{dx} = 2x + 2 = 2(x+1)$ $\quad \dfrac{dv}{dx} = -1(-\sin(3-x)) = \sin(3-x)$

$\dfrac{d(uv)}{dx} = \boxed{(x^2+2x-8)\sin(3-x) + 2(x+1)\cos(3-x)}$

Alternatively, $= -(x^2+2x-8)\sin(x-3) + 2(x+1)\cos(x-3)$

since $\cos(-x) = \cos x \quad \therefore \cos(3-x) = \cos(x-3)$

$\sin(-x) = -\sin x \quad \therefore \sin(3-x) = -\sin(x-3)$

iii) let $u = \ln 7x$ $\quad v = \sin(x^2+5)$

$\dfrac{du}{dx} = \dfrac{7}{7x} = \dfrac{1}{x}$ $\quad \dfrac{dv}{dx} = 2x\cos(x^2+5)$

$\dfrac{d\left(\frac{u}{v}\right)}{dx} = \dfrac{\dfrac{\sin(x^2+5)}{x} - 2x\ln 7x\cos(x^2+5)}{\sin^2(x^2+5)}$

$= \boxed{\dfrac{\sin(x^2+5) - 2x^2\ln 7x\cos(x^2+5)}{x\sin^2(x^2+5)}}$

iv) $\dfrac{d(\cos 4x)}{dx} = -4\sin 4x$

$\dfrac{d(\sqrt{\cos 4x})}{dx} = (-4\sin 4x)\frac{1}{2}(\cos 4x)^{-\frac{1}{2}}$

$= \boxed{-2\sin 4x(\cos 4x)^{-\frac{1}{2}}}$

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Q2

2

A curve has the equation $y = 3^x + 2^{-x}$.

Show that the gradient of the normal to the curve at the point $\left(1, \frac{7}{2}\right)$ is

$$\frac{2}{\ln 2 - 6\ln 3}$$

[4]

$m_{normal} = \dfrac{-1}{\frac{dy}{dx}}$

$y = \left(e^{\ln 3}\right)^x + \left(e^{(\ln 2)}\right)^{-x}$ $\qquad a = e^{\ln a}$

$y = e^{x\ln 3} + e^{-x\ln 2}$ $\qquad$ Chain rule

$\dfrac{dy}{dx} = (\ln 3)e^{x\ln 3} - (\ln 2)e^{-x\ln 2}$

At $x = 1$

$\dfrac{dy}{dx} = (\ln 3)e^{1\times\ln 3} - (\ln 2)\boxed{e^{-1\times\ln 2}}$

$e^{-\ln 2} = \dfrac{1}{e^{\ln 2}} = \dfrac{1}{2}$

$= 3\ln 3 - \frac{1}{2}\ln 2$

$m_{normal} = \dfrac{-1}{\frac{dy}{dx}} = \dfrac{-1}{3\ln 3 - \frac{1}{2}\ln 2} \xrightarrow{\times\frac{-2}{-2}} = \boxed{\dfrac{2}{\ln 2 - 6\ln 3}}$

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Q3

3

Find the derivative of the function $f(x) = \sin\left(\cos\left(\ln\frac{1}{x}\right)\right)$, $x > 0$.

[4]

Triple chain rule !

Start by using the chain rule to differenciate the inner function $\left(\ln\frac{1}{x}\right)$, and then work outwards.

$$\frac{d\left(\ln\frac{1}{x}\right)}{dx} = \left(-x^{-2}\right)\frac{1}{\frac{1}{x}} = -x^{-1}$$

$$\frac{d\left(\cos\left(\ln\frac{1}{x}\right)\right)}{dx} = \left(-x^{-1}\right)\left(-\sin\left(\ln\frac{1}{x}\right)\right)$$

$$= \frac{\sin\left(\ln\frac{1}{x}\right)}{x}$$

$$\frac{d\left(\sin\left(\cos\left(\ln\frac{1}{x}\right)\right)\right)}{dx} = \frac{\sin\left(\ln\frac{1}{x}\right)}{x}\cos\left(\cos\left(\ln\frac{1}{x}\right)\right)$$

$$\boxed{f'(x) = \frac{\sin\left(\ln\frac{1}{x}\right)\cos\left(\cos\left(\ln\frac{1}{x}\right)\right)}{x}}$$

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Q4a

4a

(a) Show that the derivative $y = 4^{-x^4}$ is

$$\frac{dy}{dx} = -(\ln 4)x^3 4^{1-x^4}$$

[4]

(b) Hence find the equation of the tangent to the curve at the point $\left(1, \frac{1}{4}\right)$, giving your answer in the form $y = ax + b$, where a and b are to be given as exact values.

[2]

a) $y = \left(e^{\ln 4}\right)^{-x^4} = e^{-x^4\ln 4}$

$a = e^{\ln a}$

Chain rule

$$\frac{dy}{dx} = \frac{d\left(-x^4\ln 4\right)}{dx}e^{-x^4\ln 4}$$

$$= -\boxed{4}(\ln 4)x^3\boxed{e^{-x^4\ln 4}}$$

$$4e^{-x^4\ln 4} = 4e^{\ln 4^{-x^4}} = 4\times 4^{-x^4} = 4^{1-x^4}$$

$$\therefore \boxed{\frac{dy}{dx} = -(\ln 4)x^3 4^{1-x^4}}$$

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Q4b

4b

(a) Show that the derivative $y = 4^{-x^4}$ is

$$\frac{dy}{dx} = -(\ln 4)x^3 4^{1-x^4}$$

[4]

(b) Hence find the equation of the tangent to the curve at the point $\left(1, \frac{1}{4}\right)$, giving your answer in the form $y = ax + b$, where a and b are to be given as exact values.

[2]

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b) $y - y_1 = m\,(x - x_1)$

$y_1 = \frac{1}{4}$, $x_1 = 1$

when $x = 1$,

$m = \frac{dy}{dx} = -(\ln 4)(1)^3 4^{1-1^4} = -\ln 4$

$y - \frac{1}{4} = -\ln 4\,(x - 1)$

$\boxed{y = -(\ln 4)x + \left(\ln 4 + \frac{1}{4}\right)}$

Q5a

5a

Differentiate with respect to x, simplifying your answers where possible:

(a) $\underbrace{(5 + \sin^2 3x)}_{u}\underbrace{e^{x^2-3x+2}}_{v}$

[3]

(b) $3^{\sqrt{x}}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)$

[3]

Use the product rule!

$u = 5 + \sin^2 3x$

$\;\; = 5 + \frac{1}{2}(1 - \cos 6x)$ double angle formula

$u' = \frac{1}{2}(6\sin 6x)$

$v = e^{x^2 - 3x + 2}$

$v' = (2x - 3)e^{x^2 - 3x + 2}$

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a) $\frac{d(uv)}{dx} = uv' + vu'$

$= (5 + \sin^2 3x)(2x - 3)e^{x^2 - 3x + 2}$

$+ e^{x^2 - 3x + 2}(3\sin 6x)$

$\boxed{\frac{d(uv)}{dx} = \left(e^{x^2 - 3x + 2}\right)\left((5 + \sin^2 3x)(2x - 3) + 3\sin 6x\right)}$

Q5b

5b

Differentiate with respect to x, simplifying your answers where possible:

(a) $(5+\sin^2 3x)e^{x^2-3x+2}$ [3]

(b) $\underbrace{3^{\sqrt{x}}}_{u}\underbrace{\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)}_{v}$ [3]

Product rule

$u = 3^{x^{\frac{1}{2}}} = e^{\ln 3^{x^{\frac{1}{2}}}} = e^{x^{\frac{1}{2}}\ln 3}$ $\quad a = e^{\ln a}$

$u' = \frac{1}{2}x^{-\frac{1}{2}}(\ln 3)e^{x^{\frac{1}{2}}\ln 3} = \frac{1}{2}x^{-\frac{1}{2}}(\ln 3)3^{x^{\frac{1}{2}}}$

$v = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$

$v' = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$

b) $\frac{d(uv)}{dx} = uv' + vu'$

$= 3^{x^{\frac{1}{2}}}\left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}\right) + \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\frac{1}{2}x^{-\frac{1}{2}}(\ln 3)3^{x^{\frac{1}{2}}}$

$= \frac{3^{x^{\frac{1}{2}}}}{2}\left(x^{-\frac{1}{2}} + x^{-\frac{3}{2}} + (\ln 3)x^{-\frac{1}{2}}\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\right)$

$$\boxed{\frac{d(uv)}{dx} = \frac{3^{x^{\frac{1}{2}}}}{2}\left(x^{-\frac{1}{2}} + x^{-\frac{3}{2}} + \ln 3 - (\ln 3)x^{-1}\right)}$$

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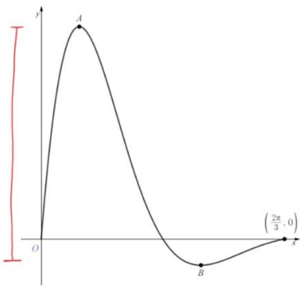
Q6

6

The diagram below shows the graph of $y = f(x)$, where $f(x)$ is the function defined by

$$f(x) = \frac{\sin 3x}{e^{2x-3}} = \frac{u}{v} \quad 0 \le x \le \frac{2\pi}{3}$$



The points A and B are maximum and minimum points, respectively.

Find the range of $f(x)$, giving your answer correct to 3 decimal places. [6]

$y_A \le f(x) \le y_B$ within the given range

$f'(x) = 0$ at stationary points

$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{vu' - uv'}{v^2}$ quotient rule

$u = \sin 3x \qquad v = e^{2x-3}$

$u' = 3\cos 3x \qquad v' = 2e^{2x-3}$

$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{\cancel{e^{2x-3}}(3\cos 3x) - (\sin 3x)(2\cancel{e^{2x-3}})}{(e^{2x-3})^{\cancel{2}}} = 0$

$3\cos 3x - 2\sin 3x = 0$

$3\cos 3x = 2\sin 3x$

$\frac{3}{2} = \tan 3x$

$x = \frac{\tan^{-1}\left(\frac{3}{2}\right)}{3}, \ \frac{\tan^{-1}\left(\frac{3}{2}\right)+\pi}{3}$

When $x = \frac{\tan^{-1}\left(\frac{3}{2}\right)}{3}$ $\quad f(x) = 8.679\ldots \rightarrow$ point A

When $x = \frac{\tan^{-1}\left(\frac{3}{2}\right)+\pi}{3}$ $\quad f(x) = -1.068\ldots \rightarrow$ point B

Range $= f(x_A) - f(x_B) = 8.679\ldots - (-1.068\ldots)$

$\boxed{\text{Range} = 9.748}$ (3dp)

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